THE ANALYST.

VOL. VII. SEPTEMBER, 1880. No. 5.

MECHANICS BY QUATERNIONS.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

STATICS.

(1). OWING to the fact that certain of the quantities treated in Mechanics possess *direction* as well as *magnitude*, and are thus in their very nature *vector* quantities, it appears that the Quaternion methods should be peculiarly fitted for dealing with mechanical problems. Such is indeed the case, and it is proposed in these papers to give an elementary quaternion treatment of the subject.

(2). According to the notation used by Hamilton a vector is always represented by some small Greek letter, except in the case of a system of 3 mutually perpendicular unit vectors which he generally represents by the letters i, j, k. Now usage has assigned to certain letters certain meanings in mechanics, such as P, Q, R, F for forces, v for velocity etc., and it seems desirable to adhere to this usage: hence it has been deemed best in these papers to indicate the fact that a letter is used as a vector by placing a subscript prime before the letter, thus:—$_{,}a$, $_{,}b$, $_{,}P$. This notation has the further advantage that a will be the tensor of $_{,}a$, P of $_{,}P$ etc., so that the use of the symbol T can be to some extent dispensed with. For convenience the primes will be omitted in the case of i, j, k used in the sense given above, and also from the letters ε_1, ε_2, ε_2 etc., used as a system of *unit* vectors in any directions, and perhaps occasionally in other cases when there is no danger of misunderstanding arising from such omission.

(3) Proceeding in the usual, though not, perhaps, the most strictly logical order, we will first discuss Statics, which treats of the laws of equilibrium; and afterwards Dynamics, which treats of the laws of motion.

The fundamental quantities of Mechanics are Space (by which is meant the path of a moving particle), Time and mass (or quantity of matter); all the other quantities to which names are given are functions of these three.

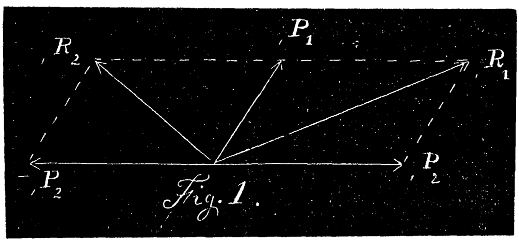
Since the path of a moving particle possesses length, and direction either constant or varying from point to point, it is evidently a vector quantity. Time and mass have no direction, and are therefore scalar quantities.

Force is that which is postulated as the cause of any change, or tendency to change, in the rate of motion. It is evidently of the nature of a vector, since the change of motion, or tendency to change, must have some direction, as well as magnitude. It will be assumed as self evident that a force may be considered as applied at any point in its line of action; whence it follows that a force ${}_{,}P$ is completely determined when we know the vector ${}_{,}\pi$ of some point in its line of action.

A body is said to be in *equilibrium* when the external forces acting on it balance, or neutralize each other.

Forces Acting on a Particle.

(4). The parallelogram of forces follows at once from the law of addition and subtraction of vectors. Thus, by Fig. 1,



$${}_{,}P_1 + {}_{,}P_2 = {}_{,}R_1 \quad \text{and}$$

$${}_{,}P_1 - {}_{,}P_2 = {}_{,}R_2.$$

Similarly we have for any number of forces

$${}_{,}P_1 + {}_{,}P_2 + {}_{,}P_3 + \&c. = \Sigma({}_{,}P) = {}_{,}R. \tag{1}$$

This equation contains in itself the propositions concerning the polygon of forces whether in a plane or in space, since ${}_{,}P_1$, ${}_{,}P_2$, &c., can have any magnitudes and directions whatever. The condition that the particle shall be in equilibrium is then evidently

$${}_{,}R = \Sigma({}_{,}P) = 0. \tag{2}$$

(5) *Equilibrium of a particle constrained to remain on a smooth curve.*—In this case it is only necessary that the resultant should have no component along the tangent to the curve at the point where the particle is situated, i. e., the resultant must be perpendicular to this tangent.

Let ${}_{,}\rho = {}_{,}\varphi(t)$, in which ${}_{,}\varphi t$ is some vector function of the scalar variable t, be the equation of a curve; then $d{}_{,}\rho = {}_{,}\varphi'(t)dt$ is a vector along the tangent. Hence the required condition may be written

$$Sd{}_{,}\rho . {}_{,}R = Sd{}_{,}\rho . \Sigma({}_{,}P) = 0, \tag{3}$$

or

$$S{}_{,}\varphi'(t)\Sigma({}_{,}P) = 0. \tag{4}$$

Suppose for instance the particle to be on a right line whose equation is

$${}_{,}\rho = {}_{,}a + {}_{,}bt.$$

Then $\frac{d_{\prime}\rho}{dt} = {}_{\prime}\varphi'(t) = {}_{\prime}b$, and $S_{\prime}b\Sigma({}_{\prime}P) = 0$ is the condition of equilibrium. As a further illustration of the use of equation (4) we will solve the following problem.

A particle is constrained to remain on the diagonal of a parallelopiped, and is acted on by three forces represented in magnitude and direction by the edges of the parallelopiped which meet at a corner not on the diagonal: determine the condition of equilibrium.

Let ${}_{\prime}a$, ${}_{\prime}b$ and ${}_{\prime}c$ be the vector edges and at the same time the three forces. Then the equation of the diagonal will be,

$$_{\prime}\rho = {}_{\prime}a + t({}_{\prime}b + {}_{\prime}c - {}_{\prime}a),$$

if we assume that it passes through the extremity of ${}_{\prime}a$. Hence

$$_{\prime}\varphi'(t) = {}_{\prime}b + {}_{\prime}c - {}_{\prime}a,$$

and by eq. (4)

$$S({}_{\prime}b + {}_{\prime}c - {}_{\prime}a)({}_{\prime}a + {}_{\prime}b + {}_{\prime}c) = 0;$$

$$\therefore\ -{}_{\prime}a^2 + {}_{\prime}b^2 + {}_{\prime}c^2 + 2S_{\prime}b_{\prime}c = 0, \text{ or } a = \sqrt{[-({}_{\prime}b + {}_{\prime}c)^2]} = T({}_{\prime}b + {}_{\prime}c).$$

(6) *Particle constrained to remain on a smooth surface.*—In this case let ${}_{\prime}\nu$ be a vector along the normal to the surface; then the condition of equilibrium is

$$V_{\prime}\nu\Sigma({}_{\prime}P) = V_{\prime}\nu_{\prime}R = 0. \qquad (5)$$

The equation of the surface may be either a scalar or vector equation. If the former, then, as the equation will be linear in $d_{\prime}\rho$ after differentiation, it may be written

$$S\psi({}_{\prime}\rho)d_{\prime}\rho = 0, \qquad (6)$$

in which $\psi({}_{\prime}\rho)$ is some vector function of ${}_{\prime}\rho$. Now $\psi({}_{\prime}\rho)$ is evidently perpendicular to $d_{\prime}\rho$ and hence $V_{\prime}\nu\psi({}_{\prime}\rho) = 0$, or we may write ${}_{\prime}\nu = \psi({}_{\prime}\rho)$.

As an example let the equation of a surface be

$$S^2{}_{\prime}a_{\prime}\rho + S_{\prime}a_{\prime}b_{\prime}\rho - S_{\prime}a_{\prime}\rho S_{\prime}b_{\prime}\rho = C;$$

differentiating

$$2S_{\prime}a_{\prime}\rho S_{\prime}ad_{\prime}\rho + S_{\prime}a_{\prime}bd_{\prime}\rho - S_{\prime}a_{\prime}\rho S_{\prime}bd_{\prime}\rho - S_{\prime}ad_{\prime}\rho S_{\prime}b_{\prime}\rho = 0,$$

or taking out $d_{\prime}\rho$,

$$S[2{}_{\prime}aS_{\prime}a_{\prime}\rho + V_{\prime}a_{\prime}b - {}_{\prime}aS_{\prime}b_{\prime}\rho - {}_{\prime}bS_{\prime}a_{\prime}\rho]d_{\prime}\rho = S\psi({}_{\prime}\rho)d_{\prime}\rho = S_{\prime}\nu d_{\prime}\rho = 0.$$

If the given equation be a vector equation it will be of the form

$$_{\prime}\rho = {}_{\prime}\varphi(t, u). \qquad (7)$$

In this case $D_t{}_{\prime}\rho$ and $D_u{}_{\prime}\rho$ will be vectors along tangents to the surface at the extremity of ${}_{\prime}\rho$, and hence we shall have

$$_{\prime}\nu = VD_t{}_{\prime}\rho D_u{}_{\prime}\rho \qquad (8)$$

as the value of ${}_{\prime}\nu$ to substitute in equation (5).

(7) We will now give several examples illustrative of the preceding formulæ.

Ex. 1. Let ABC be a triangle, and let A', B', C' be the middle points of the sides opposite to A, B and C: then if forces, represented in magnitude and direction by AA', BB', CC', act on a particle it will be in equilib'm.

Let the vector $CB = {}_{\prime}a$ and the vector $CA = {}_{\prime}b$; then the three forces, $AA' = \frac{1}{2}{}_{\prime}a - {}_{\prime}b$, $BB' = \frac{1}{2}{}_{\prime}b - {}_{\prime}a$ and $CC' = \frac{1}{2}({}_{\prime}a + {}_{\prime}b)$. The sum of these is zero, which proves the proposition.

Ex. 2. Let ε_1, ε_2, ε_3 be any three unit vectors, and $P_1\varepsilon_1$, $P_2\varepsilon_2$, $P_3\varepsilon_3$ any three forces acting along ε_1, ε_2 and ε_3: then if they are in equilibrium we have

$$P_1\varepsilon_1 + P_2\varepsilon_2 + P_3\varepsilon_3 = 0. \tag{9}$$

Operate by $S.V\varepsilon_1\varepsilon_2$, $\quad \therefore S\varepsilon_1\varepsilon_2\varepsilon_3 = 0.$ (10)

Hence the three forces must be coplanar. Next operate by $V.\varepsilon_1$, therefore $P_2V\varepsilon_1\varepsilon_2 + P_3V\varepsilon_1\varepsilon_3 = 0$, or $P_2V\varepsilon_1\varepsilon_2 = P_3V\varepsilon_3\varepsilon_1$. But since ε_1, ε_2, ε_3 are coplanar, $UV\varepsilon_1\varepsilon_2 = UV\varepsilon_3\varepsilon_1$, hence $P_2TV\varepsilon_1\varepsilon_2 = P_3TV\varepsilon_3\varepsilon_1$; as a similar relation can be found between P_1 and P_2, we have the equations

$$\frac{P_1}{TV\varepsilon_2\varepsilon_3} = \frac{P_2}{TV\varepsilon_3\varepsilon_1} = \frac{P_3}{TV\varepsilon_1\varepsilon_2} = \frac{P_1}{\sin\alpha} = \frac{P_2}{\sin\beta} = \frac{P_3}{\sin\gamma}, \tag{11}$$

if α, β, γ are the angles between ε_2 and ε_3, ε_3 and ε_1, ε_1 and ε_2 respectively.

Ex. 3. Let ε_1, ε_2, ε_3, ε_4 be any four unit vectors, and $P_1\varepsilon_1$, $P_2\varepsilon_2$, $P_3\varepsilon_3$, $P_4\varepsilon_4$ any four forces along these vectors: then for equilibrium;

$$P_1\varepsilon_1 + P_2\varepsilon_2 + P_3\varepsilon_3 + P_4\varepsilon_4 = 0. \tag{12}$$

Operate by $S.V\varepsilon_3\varepsilon_4$, $\therefore P_1S\varepsilon_3\varepsilon_4\varepsilon_1 + P_2S\varepsilon_2\varepsilon_3\varepsilon_4 = 0$, or

$$\frac{P_1}{S\varepsilon_2\varepsilon_3\varepsilon_4} = -\frac{P_2}{S\varepsilon_3\varepsilon_4\varepsilon_1}.$$

Operating by $S.V\varepsilon_4\varepsilon_1$ and $S.V\varepsilon_1\varepsilon_2$, we obtain

$$\frac{P_1}{S\varepsilon_2\varepsilon_3\varepsilon_4} = \frac{-P_2}{S\varepsilon_3\varepsilon_4\varepsilon_1} = \frac{P_3}{S\varepsilon_4\varepsilon_1\varepsilon_2} = \frac{-P_4}{S\varepsilon_1\varepsilon_2\varepsilon_3}. \tag{13}$$

These scalars are respectively proportional to the volumes of the pyramids whose edges are the vectors involved. We have also $S\varepsilon_1\varepsilon_2\varepsilon_3 \sin\alpha_4 \times \cos\beta_4$, if α_4 is the angle between ε_2 and ε_3, and β_4 that between ε_1 and $V\varepsilon_2\varepsilon_3$, and so for the others; whence

$$\frac{P_1}{\sin\alpha_1\cos\beta_1} = \frac{-P_2}{\sin\alpha_2\cos\beta_2} = \frac{P_3}{\sin\alpha_3\cos\beta_3} = \frac{-P_4}{\sin\alpha_4\cos\beta_4}. \tag{14}$$

Equations (13) and (14) are analogous to equation (11).

Ex. 4. Through a point within a sphere three chords are drawn at right angles; forces act upon this point, outwards, represented in magnitude and direction by the portions of the chords between the point and spherical surface: find the resultant.

Let the equation of the sphere be $T_{,}\rho = a$, or ${}_{,}\rho^2 = -a^2$, let i, j, k be unit vectors parallel to the chords, and let ${}_{,}\delta$ be the vector from the center of the sphere to the point. The equations of the three chords will be

$$ {}_{,}\rho = {}_{,}\delta + xi, \qquad {}_{,}\rho = {}_{,}\delta + yj, \qquad {}_{,}\rho = {}_{,}\delta + zk. $$

To find the vectors of the extremities of the chords, eliminate ${}_{,}\rho$ between the equations of the chords and sphere. Hence

$$ {}_{,}\delta^2 + 2xSi_{,}\delta - x^2 = -a^2, \therefore x = Si_{,}\delta \pm \sqrt{(S^2 i_{,}\delta + {}_{,}\delta^2 + a^2)}, $$

$$ \therefore \quad {}_{,}P_1 = i[Si_{,}\delta + \sqrt{(S^2 i_{,}\delta + {}_{,}\delta^2 + a^2)}] $$

$$ \text{and } {}_{,}P_2 = i[Si_{,}\delta - \sqrt{(S^2 i_{,}\delta + {}_{,}\delta^2 + a^2)}]. $$

By substituting j for i we obtain ${}_{,}P_3$ and ${}_{,}P_4$, and by substituting k we have ${}_{,}P_5$ and ${}_{,}P_6$. Therefore

$$ \Sigma({}_{,}P) = 2(iSi_{,}\delta + jSj_{,}\delta + kSk_{,}\delta) = -2_{,}\delta = {}_{,}R. $$

Ex. 5. Find a point on the surface

$$ \frac{S^3 i_{,}\rho}{a^3} + \frac{S^3 j_{,}\rho}{b^3} + \frac{S^3 k_{,}\rho}{c^3} + 1 = 0 $$

where a particle attracted by a force to the origin will remain at rest.

By differentiation we find

$$ {}_{,}\nu = \frac{iS^2 i_{,}\rho}{a^3} + \frac{jS^2 j_{,}\rho}{b^3} + \frac{kS^2 k_{,}\rho}{c^3}, $$

and by eq. (5) $V_{,}\nu_{,}R = -nV_{,}\nu_{,}\rho = 0$, since ${}_{,}R$ acts along ${}_{,}\rho$ towards the origin. This equation is equivalent to

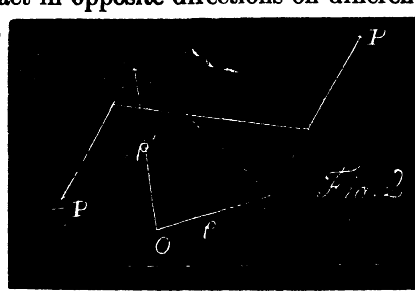
$$ {}_{,}\nu = \frac{iS^2 i_{,}\rho}{a^3} + \frac{jS^2 j_{,}\rho}{b^3} + \frac{kS^2 k_{,}\rho}{c^3} = m_{,}\rho. $$

Operating successively by $S.i$, $S.j$ and $S.k$, and also by $S._{,}\rho$, we find

$$ \frac{Si_{,}\rho}{a^3} = \frac{Sj_{,}\rho}{b^3} = \frac{Sk_{,}\rho}{c^3} = \frac{1}{{}_{,}\rho^2} = \frac{-1}{\sqrt[3]{(a^6 + b^6 + c^6)}}. $$

(8). *Moment.*—When a force acts on a particle it always produces, or tends to produce, motion in its own direction. If it be opposed by an equal force in opposite direction, the effect is neutralized.

Suppose now two equal forces to act in opposite directions on different points of a rigid body, say at the ends of a uniform bar, as in the Fig., then as before each force will tend to move its point of application straight forward in its own direction, and hence the resulting effect must be a tendency to turn the bar about its middle point. If the bar is free to move, ac-



tual rotation will be produced. If it is fastened, motion will be prevented by a pair of equal forces similar to those shown but producing an equal tendency to turn in the opposite direction. Such a pair of forces as this is called a *couple*, and all rotations are produced by the action of couples.

The tendency toward rotation of a couple is evidently proportional to the forces acting, and to the perpendicular distance between them, and hence to the product of these quantities. Thus if $_{\prime}P$ and $-_{\prime}P$ are the forces of the couple and $_{\prime}\rho$ and $_{\prime}\rho'$ vectors from some fixed point to any point in the line of direction of $_{\prime}P$ and $-_{\prime}P$; then the effect of the couple is measured by

$$_{\prime}M = V(_{\prime}\rho - _{\prime}\rho')_{\prime}P. \tag{15}$$

This quantity is called the *moment* of the couple. When the moment of a *force* about a point is spoken of, it is tacitly understood that two equal and opposite forces are inserted at the point each equal and parallel to the given force, one of which forms a couple with the given force, while the other tends to produce motion of translation at the point the same in amount that the given force would produce if acting there.

Since $_{\prime}M = V(_{\prime}\rho - _{\prime}\rho')_{\prime}P = V_{\prime}\rho_{\prime}P - V_{\prime}\rho'_{\prime}P$, it is evident that the moment of the couple is equal to the algebraic sum of the moments of the component forces about any point. It is also plain that the moment is the same about *all* points, since $_{\prime}\rho - _{\prime}\rho'$ is independent of the position of the fixed point O.

We see that the moment is a vector perpendicular to the plane of the couple, i. e., parallel to the axis of the couple, whose tensor is $PT(_{\prime}\rho - _{\prime}\rho') \times \sin\theta$, if θ is the angle between $_{\prime}\rho - _{\prime}\rho'$ and $_{\prime}P$. Thus it appears that any number of couples may be combined by adding the vectors which represent them. Hence

$$\Sigma(_{\prime}M) = \Sigma[V(_{\prime}\rho - _{\prime}\rho')_{\prime}P] = {}_{\prime}G = \text{resultant couple:}$$

and the condition of equilibrium when a number of couples act upon a body is

$$\Sigma(_{\prime}M) = \Sigma[V(_{\prime}\rho - _{\prime}\rho')_{\prime}P] = {}_{\prime}G = 0. \tag{16}$$

Let $_{\prime}\rho$ be the vector of a particle on which any forces, $_{\prime}P_1$, $_{\prime}P_2$, &c. act: then since $V_{\prime}\rho_{\prime}R = V_{\prime}\rho\Sigma(_{\prime}P) = \Sigma(V_{\prime}\rho_{\prime}P)$, it follows that the moment of the resultant about any point is equal to the sum of the moments of the forces.

If the origin be at a point of $_{\prime}R$, the vector $_{\prime}\rho$ will coincide in direction with $_{\prime}R$, so that $V_{\prime}\rho_{\prime}R = 0$: hence the moment about any point of the resultant is zero.

(9). *Parallel Forces.*—Let ε be a unit vector in any direction; then a system of parallel forces, $_{\prime}P_1$, $_{\prime}P_2$, etc., may be written $P_1\varepsilon$, $P_2\varepsilon$, etc.

For equilibrium we must have

$$\Sigma(_{\prime}P) = \Sigma(P\varepsilon) = \varepsilon\Sigma(P) = 0, \text{ i. e., } \Sigma(P) = 0; \tag{17}$$

and $\Sigma(V_{\prime}\rho_{\prime}P) = V\Sigma(P_{\prime}\rho)\varepsilon = 0$, or since $\Sigma(P_{\prime}\rho)$ is not parallel to ε,

$$\Sigma(P_{\prime}\rho) = 0. \tag{18}$$

It has been shown in the last article that the moment of the resultant of a system of forces acting at a point is equal to the sum of the moments of the forces. This is equally true of a system of parallel forces ; for let some point of each force in the first system be fixed and then let the common point of the forces move off in some direction to infinity. The law holds up to the limit, and then the forces have become parallel. Hence we have, if ${}_{\prime}\rho_0$ is a vector to some point of the resultant,

$$V_{\prime}\rho_{0\prime}R = V\Sigma(P_{\prime}\rho)\varepsilon = RV_{\prime}\rho_0\varepsilon, \text{ or}$$

$$V[R_{\prime}\rho_0 - \Sigma(P_{\prime}\rho)]\varepsilon = 0. \tag{19}$$

Now, as ε is not in general parallel to the vector in parentheses, we must have $R_{\prime}\rho_0 = \Sigma(P_{\prime}\rho)$ or ${}_{\prime}\rho_0 = \Sigma(P_{\prime}\rho) \div R$; (20)
thus we have determined the position of the resultant. As eq .(20) is inde pendent of ε, it is plain that the value of ${}_{\prime}\rho_0$ remains unchanged whatever direction the system of parallel forces may have, so long as ${}_{\prime}P_1$, ${}_{\prime}P_2$, etc., pass through the extremities of ${}_{\prime}\rho_1$, ${}_{\prime}\rho_2$, etc., the latter being supposed const.

Thus if w_1, w_2, etc. be the weights of a system of heavy particles at the extremities of ${}_{\prime}\rho_1$, ${}_{\prime}\rho_2$ etc.; then the vector of the center of gravity of the system will be

$${}_{\prime}\rho_0 = \frac{\Sigma(w_{\prime}\rho)}{\Sigma(w)}. \tag{21}$$

If in (20) $R = \Sigma(P) = 0$, then ${}_{\prime}\rho_0 = \infty$, i. e., when the system reduces to a couple the resultant is a force $= 0$ applied at ∞. If $\Sigma(P_{\prime}\rho) = 0$, ${}_{\prime}\rho_0 = 0$, and the origin is at a point of the resultant, which agrees with the last statement of Art. 8, since if $\Sigma(P_{\prime}\rho) = 0$ the moment about the origin is nothing by eq. (18). In case of equilibrium $\Sigma(P_{\prime}\rho) = R = 0$, and ${}_{\prime}\rho_0$ becomes indeterminate.

(10). *Resultant of any system of forces acting on a rigid body.*—Let ${}_{\prime}\pi$ be the vector of some fixed point, and, as before, ${}_{\prime}\rho_1$, ${}_{\prime}\rho_2$ etc., vectors from the origin to points upon the lines of action of ${}_{\prime}P_1$, ${}_{\prime}P_2$ etc. Suppose that at the extremity of ${}_{\prime}\pi$ there act the forces ${}_{\prime}P_1$ and $-{}_{\prime}P_1$, ${}_{\prime}P_2$ and $-{}_{\prime}P_2$ etc.: these will balance each other, and therefore will not affect the system. Now ${}_{\prime}P_1$ at ${}_{\prime}\rho$ will form a couple with $-{}_{\prime}P_1$ at ${}_{\prime}\pi$ whose moment will be ${}_{\prime}M = V({}_{\prime}\rho_1 - {}_{\prime}\pi)_{\prime}P_1$, while there will remain the force ${}_{\prime}P_1$ acting at ${}_{\prime}\pi$, and the same statement holds for each of the other forces. Hence the whole system of forces will be reduced to a single force ${}_{\prime}R = \Sigma({}_{\prime}P)$ acting at ${}_{\prime}\pi$, and a couple ${}_{\prime}G = \Sigma({}_{\prime}M) = \Sigma[V({}_{\prime}\rho - {}_{\prime}\pi)_{\prime}P] = V\Sigma[({}_{\prime}\rho - {}_{\prime}\pi)_{\prime}P]$

$$= V\Sigma({}_{\prime}\rho_{\prime}P) - V_{\prime}\pi\Sigma({}_{\prime}P) = V\Sigma({}_{\prime}\rho_{\prime}P) - V_{\prime}\pi_{\prime}R.$$

The quantity $V\Sigma({}_{\prime}\rho_{\prime}P)$ being independent of ${}_{\prime}\pi$ will be the same for all values of ${}_{\prime}\pi$ so long as the position of the origin is unchanged. Let us call it ${}_{\prime}G_0$, so that ${}_{\prime}G = {}_{\prime}G_0 - V_{\prime}\pi_{\prime}R$.

The general conditions of equilibrium then are

$$ {}_{\prime}R = \Sigma({}_{\prime}P) = 0, \tag{22} $$

$$ {}_{\prime}G = \Sigma({}_{\prime}M) = \Sigma[V({}_{\prime}\rho - {}_{\prime}\pi),{}_{\prime}P] = {}_{\prime}G_0 - V_{\prime}\pi_{\prime}R = 0. \tag{23} $$

But when ${}_{\prime}R = 0$, ${}_{\prime}G = {}_{\prime}G_0$, so that (23) becomes ${}_{\prime}G = {}_{\prime}G_0 = V\Sigma({}_{\prime}\rho_{\prime}P) = 0. \quad (23_a)$

(11). Before proceeding to develope certain conditions and relations from the preceding formulæ we will give a few examples illustrative of their use.

Ex. 1. Show that if a given system of forces be reduced in any manner to *two* forces which do not meet and are not parallel, the volume of the tetrahedron of which these two forces form opposite vector edges is constant.

Let the two forces be ${}_{\prime}P$ and ${}_{\prime}P'$, and let ${}_{\prime}\rho$ and ${}_{\prime}\rho'$ be vectors to points in their respective lines of action. Then since we have a given system

$$ {}_{\prime}P + {}_{\prime}P' = {}_{\prime}R = \text{a constant vector}, $$

and $\quad V_{\prime}\rho_{\prime}P + V_{\prime}\rho'{}_{\prime}P' = {}_{\prime}G = \text{a constant vector}.$

Operate by $S.{}_{\prime}P$ and $S.{}_{\prime}P'$, $\therefore$ $S_{\prime}P_{\prime}\rho_{\prime}P' = S_{\prime}P_{\prime}G$, and $S_{\prime}P'{}_{\prime}\rho_{\prime}P = S_{\prime}P'{}_{\prime}G$. Adding these two equations we have

$$ S_{\prime}P_{\prime}P'({}_{\prime}\rho - {}_{\prime}\rho') = S({}_{\prime}P + {}_{\prime}P'),{}_{\prime}G = S_{\prime}R_{\prime}G = \text{constant}, $$

which proves the proposition.

Ex. 2. Show that if forces act at the middle points of the sides of any plane polygon, the forces being perpendicular and proportional to the respective sides, and directed either all inward or all outward, they will be in equilibrium.

Let the sides of the polygon taken around in order be ${}_{\prime}\alpha_1$, ${}_{\prime}\alpha_2$, etc., and let ε be a unit vector perpendicular to the plane of the polygon. Then the forces may be taken as $\varepsilon_{\prime}\alpha_1$, $\varepsilon_{\prime}\alpha_2$ etc. Now as

$$ {}_{\prime}\alpha_1 + {}_{\prime}\alpha_2 + \dots + {}_{\prime}\alpha_n = \Sigma({}_{\prime}\alpha) = 0, $$

it follows that

$$ \Sigma({}_{\prime}P) = \varepsilon\Sigma({}_{\prime}\alpha) = 0. $$

Next taking moments about O we have ${}_{\prime}\pi = 0$ in eq. (23), and therefore

$$ {}_{\prime}G = \Sigma(V_{\prime}\rho_{\prime}P) = \tfrac{1}{2}V_{\prime}\alpha_1\varepsilon_{\prime}\alpha_1 + V({}_{\prime}\alpha_1 + \tfrac{1}{2}{}_{\prime}\alpha_2)\varepsilon\alpha_2 + \dots $$
$$ \dots + V({}_{\prime}\alpha_1 + {}_{\prime}\alpha_2 + {}_{\prime}\alpha_3 + \dots + {}_{\prime}\alpha_{n-1} + \tfrac{1}{2}{}_{\prime}\alpha_n)\varepsilon_{\prime}\alpha_n. $$

But $V_{\prime}\alpha_1\varepsilon_{\prime}\alpha_1 = -\varepsilon_{\prime}\alpha_1^2$, etc., so that

$$ -2{}_{\prime}G = \varepsilon_{\prime}\alpha_1^2 + \varepsilon_{\prime}\alpha_2^2 + \dots + \varepsilon_{\prime}\alpha_n + 2\varepsilon(S_{\prime}\alpha_1{}_{\prime}\alpha_2 + S_{\prime}\alpha_1{}_{\prime}\alpha_3 + \dots + S_{\prime}\alpha_1{}_{\prime}\alpha_n) $$
$$ + 2\varepsilon(S_{\prime}\alpha_2{}_{\prime}\alpha_3 + S_{\prime}\alpha_2{}_{\prime}\alpha_4 + \dots + S_{\prime}\alpha_2{}_{\prime}\alpha_n) + \dots + 2\varepsilon S_{\prime}\alpha_{n-1}{}_{\prime}\alpha_n $$
$$ = \varepsilon[\Sigma({}_{\prime}\alpha)]^2 = 0. $$